On Some Advantages of the Hyper Geometric Representation of the Associated Legendre Functions of the First and Second Kinds

Haniyah A. M. Saed

Mathematics Department, Faculty of Science, Sirte University, Sirte, Libya, Tel: 0925286621, E-mail: hanyahamdin23@yahoo.com

Abstract

The linear and the quadratic transformations of the hyper geometric function are proven very useful in making various transformations and carrying out the analytic continuation of hyper geometric function into any part of the complex z-plane cut along the real axis from the point $z= +1$ to the point $z= +\infty$. Here we shall represent the associated Legendre functions (or spherical functions) of the first and second kinds in terms of the hyper geometric function to gain their analytic continuation into any part of the complex z-plane. Furthermore, the hyper geometric representation enables us to develop the theory of spherical functions by implementing the general theory of the hyper geometric function. Obtaining the hyper geometric representation of such functions by means of linear and quadratic transformations is more general and less complicated than the Euler’s integral representation which is restricted to certain constraints to the values of the parameters of the hyper geometric function that are essential to make use of the integral definition of the Beta function.

Keywords: Hyper geometric function; Hyper geometric series; Associated legendre functions of the first and second kinds; Spherical functions; Linear and quadratic transformations of the hyper geometric function

Introduction

Motivated by the great importance of special functions in general and the Legendre functions in particular, here we shall relate the associated Legendre functions to the so called Gauss hyper geometric function (Abramowitz and Stegun 1968; Andrews et al. 1999; Laham and Abdallah 1996; Rainville 1960; Lebedev 1965). The hyper geometric function was introduced by Euler and then studied thoroughly by Gauss (Laham and Abdallah 1996) and plays an important role in mathematical analysis and its applications such as conformal mapping of triangular domains bounded by line segments or circular arcs (Lebedev 1965). The Legendre functions have been discovered by Laplace and Legendre as early in the 18th century and they are connected with many problems of mathematical physics, in the potential theory for spheroidal, toroidal and other coordinates (Hobson 1955). The associated Legendre functions of the first and second kinds (Kuipers 1957; Meulenbeld and Virchenko 1987) possess high importance in variety of applications to problems of physics, quantum mechanics, and engineering. Many algebraic and transcendental functions that appear in problems of mathematical physics can be expressed in terms of the hyper geometric function, thus the theory of these functions can be considered as a special case of the general theory of the hyper geometric functions (Lebedev 1965). The hyper geometric representation of the associated Legendre functions has the great advantage of obtaining the analytic continuation...
of these functions into any part of the complex z-plane (Lebedev 1965). In turn this should allow variety of applications for such functions. The hyper geometric representation of any function can be achieved with the aid of the so called linear as well as nonlinear transformations on the independent variable for the hyper geometric function (Lebedev 1965). Such transformations were derived extensively in an elaborate manner in (Erdelyi 1964; Magnus and Oberhettinger 1953) and references therein. The linear transformations consist of all the following fractional linear form.

\[ z' = \frac{az + b}{cz + d}, \quad a, b, c, d \in \mathbb{C}. \]

Since the core idea of deriving the linear transformations comes from the theory of the twenty-four solutions of the hyper geometric differential equation discovered by Kumar in 1836, sometimes the linear transformations are called by Kumar’s relations (Rainville 1964). The nonlinear transformations contain expressions like

\[ (1 - \sqrt{1 - z})/2, \quad (1 - \sqrt{1 - z})/(1 + \sqrt{1 - z}), \quad 1/(1 - z)^2, \quad z^2, \ldots \]

Which are known as the quadratic transformations of the hyper geometric function. In fact the theory of quadratic transformations of the hyper geometric function is an old topic and can be traced back to Gauss, Kummer, and Goursat (Lebedev 1965). For an extensive list of such transformations the reader is referred to the references (Abramowitz and Stegun 1968; Lebedev 1965; Rainville 1964). Erdelyi et al. 1953-55). Conceptually, both kinds of transformations are proven very useful in making various transformations and known as Euler’s transformations of the hyper geometric function (Rainville 1964). It is worth mentioning that the linear and nonlinear transformations are among the most important relations in the theory of hyper geometric function. However there are other approaches to investigate the properties of the hyper geometric functions and obtain their analytic continuation. For instance, Hobson (1955) investigated properties of the associated Legendre functions by means of contour integrals defined in terms of Pochhammer symbol (Abramowitz and Stegun 1968) and Jordan double contour integrals (Verchenko and Rumiantseva 2008). Such an approach has the privilege of being away from the convergence issue of infinite series. The main results in the theory of the generalized associated Legendre functions have been established by Meulenbeldin Kuipers and Meulenbeld (1957). Also Verchenko and Rumiantseva, (2008) considered the generalized hyper geometric function (Verchenko 1999; Verchenko et al. 2001; Raoa and Shukla 2013; Malovichko 1976) to gain an integral representation of the generalized associated Legendre functions of both kinds (Kuipers and Meulenbeld 1957; Verchenko and Rumiantseva 2008; Verchenko and Fedotova 2001). Verchenko et al. implemented the integral form of the generalized (in the sense of Wright (Verchenko and Fedotova 2001)) hyper geometric function which is defined in terms of the so called Fox-Wright functions (Verchenko and Fedotova 2003; Wright 1935) obtaining an integral form of the generalized associated Legendre functions. A further generalized hyper geometric k-functions is defined and some properties are established in (Rahman et al. 2016; Miller 2003) using a special case of Wright hyper geometric function. Since the classical Gaussian hyper geometric function and the associated Legendre functions are respectively, special cases of the generalized hyper geometric function and the generalized associated Legendre functions, one could claim that Verchenko and Rumiantseva (2008) presented a more general approach using the generalization concept of such functions. The Euler’s integral representation of the hyper geometric function (Rainville 1964) can be obtained using the integral definition of the Beta function (Abramowitz and Stegun 1968). Such an integral form of the hyper geometric function is less general because is often restricted to certain constraints on the values of the parameters of the hyper geometric function which are essential to make use of the integral definition of the Beta function. We will consider the hyper geometric representation of the considered functions only by means of linear and quadratic transformations of the hyper geometric function, referring the reader elsewhere for integral representations (Lebedev 1965).
This paper is structured as follows: in section one; we briefly set up the notations for the hyper geometric function. After obtaining the solutions of the associated Legendre differential equation in section two, respectively, a hyper geometric representation of the associated Legendre functions of the first and second kinds are presented in sections three and four. Finally, a discussion and conclusion is drawn on the hyper geometric representation of the associated Legendre functions in sections five and six.

**Overview on the Gauss Hyper Geometric Function**

In this section we shall introduce some notations that are used in this paper. Consider the series

\[
\sum_{n=0}^\infty \frac{\alpha(\alpha+1)K(\alpha+n-1)\beta(\beta+1)K(\beta+n-1)}{\gamma(\gamma+1)K(\gamma+n-1)} \frac{z^n}{n!}
\]

Where \(z\) is a complex variable \(\alpha\) or \(\beta\) and \(\gamma\) are parameters, which can take arbitrary real or complex values provided that \(\gamma \neq 0, -1, -2, K\). If we let \(\alpha = 1\) and \(\beta = \gamma\), then we get the elementary geometric series \(\sum_{n=0}^\infty \frac{z^n}{n!}\). The series (1) is called the Gauss hyper geometric series, which has great importance in mathematical analysis and its applications. Using the generalized factorial function \((Abramowitz and Stegun 1968)\) or Pochhammer symbol \((a)_n\) defined as

\[
(a)_n = \prod_{k=0}^{n} (a + k - 1), \quad (a)_0 = 1, \quad a \neq 0,
\]

We can simplify the series (1) in the form

\[
\sum_{n=0}^\infty \frac{(\alpha)_n (\beta)_n}{(\gamma)_n n!} \frac{z^n}{n!}
\]

This series can be written in terms of the gamma function using the following relation between the Pochhammer symbol and the gamma function defined as

\[
(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}, \quad a \neq 0, \quad n=0,1,2,...
\]

Hence, one has

\[
\sum_{n=0}^\infty \frac{(\alpha)_n (\beta)_n}{(\gamma)_n n!} \frac{z^n}{n!} = \frac{\Gamma(\gamma)}{\Gamma(\alpha) \Gamma(\beta)} \sum_{n=0}^\infty \frac{\Gamma(\alpha+n) \Gamma(\beta+n)}{\Gamma(\gamma+n) \Gamma(n+1)} \frac{z^n}{n!}.
\]

By using the ratio test, it can be easily proved that the radius of convergence of the hyper geometric series (1) is unity \(|z| < 1\), except when the parameters \(\alpha\) or \(\beta\) is zero or a negative integer, in which case the series (1) terminates and turns to a polynomial where the convergence has no sense. Also using the Gauss test, it can be shown that the hyper geometric series converges absolutely for \(|z| = 1\) provided \(\Re(\gamma - \alpha - \beta) > 0\) that \((Rainville 1960; Lebedev 1965; Rainville 1964)\). We shall denote the convergent hyper geometric series by the notation \(\frac{\Gamma(\gamma)}{\Gamma(\alpha) \Gamma(\beta)} \sum_{n=0}^\infty \frac{\Gamma(\alpha+n) \Gamma(\beta+n)}{\Gamma(\gamma+n) \Gamma(n+1)} \frac{z^n}{n!}\) by the notation that is,

\[
F(\alpha, \beta; \gamma; z) = \sum_{n=0}^\infty \frac{(\alpha)_n (\beta)_n}{(\gamma)_n n!} \frac{z^n}{n!}, \quad |z| < 1, \quad \gamma \neq 0, -1, -2,...
\]

That is there exists a complex function which is analytic in the complex \(z\)-plane cut along the real axis from the point \(z = 1\) to the point \(z = \infty\) and coincides with \(F(\alpha, \beta; \gamma; z)\) inside the unit disc. Moreover \(F(\alpha, \beta; \gamma; z)\) is an analytic function of its parameters \(\alpha\) or \(\beta\) and a meromorphic function of its parameter \(\gamma\), with simple poles at the points \(\gamma=0, -1, -2 \ldots\) \((Lebedev 1965)\).

**The Associated Legendre Differential Equation**

The linear, second-order, homogeneous, spherical harmonic differential equation

\[
(1-z^2)\frac{d^2y}{dz^2} - 2z \frac{dy}{dz} + \left[\mu(\mu+1) - \frac{\nu^2}{2(1-z^2)} - \frac{\eta^2}{2(1+z^2)}\right]y(z) = 0,
\]
is called the generalized associated Legendre differential equation where $z$ is a complex variable. The parameters $\nu$ and $\mu$ are arbitrary complex constants and called the order and the degree of the corresponding generalized associated Legendre functions, respectively. This equation arises from separation of variables in solving Laplace equation $\Delta y = 0$ in spherical coordinates (Kuipers and Meulenbeld 1957) obtained solutions of the generalized associated Legendre differential equation which are valid for unrestricted values of the parameters $\mu$, $\eta$ and $\nu$. Their results have been expressed in terms of contour integrals and the hypergeometric functions. For some applications, it is often necessary to solve the associated Legendre differential equation for real values of the variable $z$ in the interval $(-1, +1)$, that is for $z = \cos \theta$, and for integral values of the parameters $\mu$ and $\nu$. Hobson (1896) presented definitions of the associated Legendre functions for unrestricted values of the parameters $\nu$, $\mu$ and the argument $z$ by means of contour integrals. The generalized associated Legendre differential equation can be reduced to the following classical associated Legendre differential equation as we set the parameters $\eta=\nu=n$, thus one has for arbitrary $\mu$ and nonnegative integral values of $n$,

$$
(1-z^2)\frac{d^2y}{dz^2} - 2z \frac{dy}{dz} + \left[ \mu(\mu+1) - \frac{n^2}{1-z^2} \right] y(z) = 0, \quad n = 0, 1, 2, \ldots \quad (4)
$$

This equation yields the ordinary Legendre differential equation as $n=0$ and it is well known in mathematical physics to solve boundary value problems of potential theory, geodesy and quantum mechanics. To solve equation (4), we assume that the variable $z$ belongs to the complex $z$-plane cut along the real axis from the point $z = -\infty$ to the point $z = 1$, and introduce the following gauge transformation in terms of the new function $w$ which is related to the function $y$ by the formula (Lebedev 1965; Beals 2010).

$$
w(z) = (z^2 - 1)^{\frac{n}{2}} y(z).
$$

The derivatives of the function $y$ are obtained as

$$
y' = (z^2 - 1)^{\frac{n}{2}} w' + nzw \left( z^2 - 1 \right)^{\frac{n}{2}-1},
y'' = (z^2 - 1)^{\frac{n}{2}} w'' + 2nz \left( z^2 - 1 \right)^{\frac{n}{2}-1} w' + nw \left( z^2 - 1 \right)^{\frac{n}{2}-1} \frac{z^2(n-1)-1}{z^2-1}.
$$

Substituting these derivatives in equation (4) leads to

$$
(1-z^2)w'' - 2(n+1)zw' + (\mu-n)(\mu+n+1)w = 0. \quad (5)
$$

Now if $u$ be a solution of the ordinary Legendre differential equation, then one has

$$
(1-z^2)u'' - 2zu' + \mu(\mu+1)u = 0
$$

Differentiating this equation $n$ times with respect to, and letting $w = \frac{d^nu}{dz^n}$ leads to

$$
(1-z^2)w'' - 2(n+1)zw' + (\mu-n)(\mu+n+1)w = 0
$$

Hence we showed that the function is a solution of equation (5). It follows that the solutions of equation (4) are obtained as the following

$$
y_1 = (z^2 - 1)^{\frac{n}{2}} \frac{d^n u_1}{dz^n}, \quad \text{and} \quad y_2 = (z^2 - 1)^{\frac{n}{2}} \frac{d^n u_2}{dz^n}.
$$

Where $u_1 = P_\nu(z)$, and $u_2 = Q_\nu(z)$ are solutions of the ordinary Legendre differential equation. The single-valued functions $y_1(z)$ and $y_2(z)$ are denoted by $P_\nu'(z)$ and $Q_\nu'(z)$ respectively, and are called the associated Legendre functions (or spherical functions) of the first and second kinds, respectively, of order $\mu$ and degree $n$ that is,
\[ P_\mu^n(z) = (z^2 - 1)^{\frac{n}{2}} \frac{d^n P_\mu(z)}{dz^n}, \quad (6) \]

\[ Q_\mu^n(z) = (z^2 - 1)^{\frac{n}{2}} \frac{d^n Q_\mu(z)}{dz^n}, \quad (7) \]

Sometimes equations (6) and (7) are called by Hobson definition of the associated Legendre functions (Whittaker and Watson 1952; Hobson and Barnes 1908). For general values of \( \mu \) the Legendre functions \( P_\mu(z) \) and \( Q_\mu(z) \) of the first and second kinds, respectively, are analytic in the complex \( z \)-plane cut along the segments \((-\infty, -1] \) and \((-\infty, 1]\), respectively. It follows that from equations (6) and (7) that \( P_\mu^n(z) \) and \( Q_\mu^n(z) \) are entire functions of the variable \( z \) in the complex \( z \)-plane cut along the the real axis from the point \( z = -\infty \) to the point \( z = +1 \) (Lebedev 1965). We already know that the general solution of the ordinary Legendre differential equation defined as

\[ u_\mu(z) = A P_\mu(z) + B Q_\mu(z), \quad (8) \]

Where \( A \) and \( B \) are constants. Differentiating relation (8) \( n \) times and multiplying by the factor

\[ (z^2 - 1)^{\frac{n}{2}} \] leads to

\[ (z^2 - 1)^{\frac{n}{2}} \frac{d^n u_\mu}{dz^n} = A \left( z^2 - 1 \right)^{\frac{n}{2}} \frac{d^n P_\mu(z)}{dz^n} + B \left( z^2 - 1 \right)^{\frac{n}{2}} \frac{d^n Q_\mu(z)}{dz^n} \]

Hence the general solution of equation (4) is obtained as

\[ u(z) = A P_\mu(z) + B Q_\mu(z) \]

Some of the associated Legendre functions of the first and second kinds are obtained as,

\[ P_\mu^0(z) = 1, \quad P_\mu^1(z) = -(z^2 - 1)^{\frac{1}{2}}, \quad Q_\mu^0(z) = \frac{\ln\left(\frac{z+1}{z-1}\right)}{2}, \quad Q_\mu^1(z) = \frac{1}{2} \left[ \ln\left(\frac{z+1}{z-1}\right) + \frac{z}{z^2-1} \right] \]

The associated Legendre functions are defined in restricted regions of the complex \( z \)-plane for \( z \in (-\infty, 1]\), next we show how they can be continued analytically to other regions by obtaining their hyper geometric representation.

**Hyper Geometric Forms of the First Kind Associated Legendre Functions**

\( P_\mu^0(z) \)

In this section we shall represent the first kind associated Legendre functions \( P_\mu^0(z) \) in terms of the hyper geometric function to carry out their analytic continuation into different parts of the complex \( z \)-plane (Beals and Wong 2010). This can be achieved with the aid of some linear as well as quintic transformations of the hyper geometric functions. Starting by substituting the Murphy’s expression \( P_\mu^0(z) \) of given in (Whittaker and Watson 1952) as a hyper geometric function into equation (6) to obtain

\[ P_\mu^0(z) = (z^2 - 1)^{\frac{1}{2}} \frac{d^2 P_\mu(z)}{dz^2}, \quad (9) \]

The n-fold differentiation of the hyper geometric function can be computed using the following relation (Abramowitz and Stegun 1968; Lebedev 1965).

\[ \frac{d^n}{dz^n} F(\alpha, \beta; \gamma; z) = \frac{(\alpha)_n (\beta)_n}{(\gamma)_n} F(\alpha + n, \beta + n; \gamma + n; z) \]

So, equation (9) becomes

\[ P_\mu^0(z) = \frac{(z^2 - 1)^{\frac{3}{2}}}{2} (-1)^{n-1} (\frac{\mu}{1})_n F\left(-\mu, n + \mu + 1; n + 1; \frac{1-z}{2}\right). \quad (10) \]
We use the relation (3) to simplify expressions in equation (10), hence one has

\[
(\mu + 1)_n = \frac{\Gamma(\mu + n + 1)}{\Gamma(\mu + 1)}, \quad \text{and} \quad (-\mu)_n = \frac{\Gamma(n - \mu)}{\Gamma(-\mu)}
\]

But \( \Gamma(n - \mu) = \Gamma(-\mu + n) = (-\mu + n - 1)(-\mu + n - 2)\cdots (-\mu) \Gamma(-\mu) \), \( n = 2, 3, \ldots \), \( (-\mu)_n \Gamma(-\mu) \)

So, one has

\[
\frac{\Gamma(n - \mu)}{\Gamma(-\mu)} = \frac{(-1)^n (\mu - n + 1)(\mu - n + 2)\cdots \Gamma(\mu)}{\Gamma(\mu)} \quad \text{and} \quad \frac{\Gamma(-\mu)}{\Gamma(n - \mu)} = \frac{(-1)^n (\mu - n + 1)(\mu - n + 2)\cdots \Gamma(-\mu)}{\Gamma(-\mu)}
\]

But, \( \Gamma(\mu) = (\mu - 1)(\mu - 2)\cdots (\mu - n + 1) \Gamma(-\mu + n) = n, 3, \ldots \)

So from equation (11), one has

\[
(-\mu)_n = \frac{\Gamma(n - \mu)}{\Gamma(-\mu)} = \frac{(-1)^n \Gamma(\mu + 1)}{\Gamma(\mu - n + 1)}
\]

Substituting this in equation (10) leads to the hyper geometric form of the associated Legendre functions of the first kind as

\[
P^\mu_n(z) = \frac{\Gamma(\mu + n + 1)}{2^n \Gamma(\mu - n + 1) \Gamma(n + 1)} \left(\frac{z^2 - 1}{\Gamma(\mu + 1)}\right)^{\mu/2} F\left(\begin{array}{c}
-\mu, n + \mu + 1 ; n + 1 ; \frac{1-z}{2} \end{array} \right), \quad |z| < 2
\] (12)

Also we can apply another transformation for the hyper geometric representation \( P^\mu_n(z) \) of to obtain different forms \( P^\mu_n(z) \) of defined in different regions in the complex \( z \)-plane. So if we apply the following transformation which is due to Euler (Lebedev 1965; Beals and Wong 2010).

\[
F\left(\begin{array}{c}
\alpha, -\beta, \gamma + \beta, \gamma ; z \end{array} \right) = (1-z)^{-\beta} F\left(\begin{array}{c}
\gamma - \alpha, \gamma - \beta, \gamma ; z \end{array} \right), \quad |z| < 2
\] (13)

to the hyper geometric function on the right-hand side of equation (12), then one has

\[
P^\mu_n(z) = \frac{\Gamma(\mu + n + 1)}{\Gamma(\mu - n + 1) \Gamma(n + 1)} \left(\frac{z^2 - 1}{\Gamma(\mu + 1)}\right)^{\mu/2} F\left(\begin{array}{c}
\mu + 1, -\mu, n + 1 ; 1-z/2 \end{array} \right), \quad |z| < 2
\] (14)

Furthermore if we apply the following quadratic transformation (Lebedev 1965; Rainville 1964).

\[
F\left(\begin{array}{c}
\alpha, \beta, \gamma ; z \end{array} \right) = F\left(\begin{array}{c}
\alpha + 0.5, \beta + 0.5, \gamma + 0.5 ; 1-\sqrt{1-z} \end{array} \right), \quad |1-\sqrt{1-z}| < 2
\] (15)

to the hyper geometric function on the right-hand side of equation (12), then one has

\[
P^\mu_n(z) = \frac{\Gamma(n + 1)}{\Gamma(\mu - n + 1) \Gamma(n + 1)} \left(\frac{z^2 - 1}{\Gamma(\mu + 1)}\right)^{\mu/2} F\left(\begin{array}{c}
\frac{n - \mu}{2}, \frac{n + \mu + 1}{2}, n + 1 ; 1-z^2 \end{array} \right), \quad |1-z^2| < 1, \quad n = 1, 2, \ldots
\] (16)

In which we set the parameters in equation (15) as \( \alpha = \frac{n - \mu}{2}, \beta = \frac{n + \mu + 1}{2} \), and \( \alpha + \beta + \frac{1}{2} = n + 1 \).

Further hyper geometric representations of the first kind associated Legendre functions can be found in the references (Abramowitz and Stegun 1968; Rainville 1960; Lebedev 1965; Rainville 1964; Erdelyi et al. 1953-1955; Gradshteyn and Ryzhik 2007). Primarily, we are not interested in deriving the hyper geometric forms of the associated Legendre functions rather we aim to show that the transformations approach is simple and convenient because it only calls some transformations and then carry out appropriate passages to the limit.
Hyper Geometric Form of the Second Kind Associated Legendre functions $Q_n^\mu(z)$.

In this section we shall deduce the hyper geometric form of the second kind associated Legendre functions $Q_n^\mu(z)$ (Lebedev 1965; Beals and Wong 2010). Starting from the hyper geometric form of the Legendre functions of the second kind obtained as (Abramowitz and Stegun 1968).

$$Q_n^\mu(z) = \frac{\sqrt{\pi}\Gamma(\mu+1)}{\Gamma\left(\mu + \frac{3}{2}\right) (2z)^{\mu+1}} F\left(\frac{\mu+1}{2}, \frac{\mu+1}{2}; \frac{\mu+3}{2}, \frac{1}{2}; \frac{1}{z^2}\right)$$

$$= \frac{\sqrt{\pi}\Gamma(\mu+1)}{\Gamma\left(\mu + \frac{3}{2}\right) 2^{\mu+1}} \sum_{k=0}^{\infty} \frac{\left(\frac{\mu+1}{2}\right)_k \left(\frac{\mu+1}{2}\right)_k}{k!} z^{-2k}$$

Using the relation (3) leads to

$$Q_n^\mu(z) = \frac{\sqrt{\pi}\Gamma(\mu+1)}{\Gamma\left(\mu + \frac{3}{2}\right) 2^{\mu+1}} \sum_{k=0}^{\infty} \frac{\left(\frac{\mu+1}{2}\right)_k \left(\frac{\mu+1}{2}\right)_k}{k!} z^{-2k}$$

$$= \frac{\sqrt{\pi} \Gamma(\mu+1) \Gamma\left(\mu + \frac{3}{2}\right)}{\Gamma\left(\mu + \frac{3}{2}\right) 2^{\mu+1} \Gamma\left(\mu + \frac{1}{2}\right)} \sum_{k=0}^{\infty} \frac{\left(\frac{\mu+1}{2}\right)_k \left(\frac{\mu+1}{2}\right)_k}{k!} z^{-2k}.$$

Use the gamma function of the argument $2z$ (Abramowitz and Stegun 1968), by taking $z = \frac{\mu+1}{2}$ to obtain

$$\frac{\sqrt{\pi}\Gamma(\mu+1)}{\Gamma\left(\mu + \frac{3}{2}\right) 2^{\mu+1}} \sum_{k=0}^{\infty} \frac{\left(\frac{\mu+1}{2}\right)_k \left(\frac{\mu+1}{2}\right)_k}{k!} z^{-2k} = \frac{2^n \Gamma\left(\frac{\mu+1}{2}\right) \Gamma\left(\frac{\mu+1}{2}\right) \Gamma\left(\mu + \frac{3}{2}\right)}{\Gamma\left(\mu + \frac{3}{2}\right) 2^{\mu+1} \Gamma\left(\mu + \frac{1}{2}\right)} = \frac{1}{2^n}.$$

Hence, one has

$$Q_n^\mu(z) = \frac{1}{2^n} \sum_{k=0}^{\infty} \frac{\left(\frac{\mu+1}{2}\right)_k \left(\frac{\mu+1}{2}\right)_k}{k!} z^{-2k}.$$  (17)

Differentiate $Q_n^\mu(z)$ given by equation (17) $n$ times with respect to $z$ to obtain

$$\frac{d^n}{dz^n} Q_n^\mu(z) = \frac{1}{2^n} \sum_{k=0}^{\infty} \frac{\left(\frac{\mu+1}{2}\right)_k \left(\frac{\mu+1}{2}\right)_k}{k!} \frac{d^n}{dz^n} z^{-2k}.$$  (18)

Use the gamma function of the argument $2z$ (Abramowitz and Stegun 1968), to get

$$\frac{\Gamma(2k + \mu + n + 1)}{\Gamma(2k + \mu + 1)} = \frac{\Gamma\left(2k + \mu + n + 1\right)}{\Gamma\left(2k + \mu + 1\right)} \frac{\Gamma\left(2k + \mu + n + 1\right)}{\Gamma\left(2k + \mu + n + 1\right)} = \frac{\Gamma\left(2k + \mu + n + 1\right)}{\Gamma\left(2k + \mu + 1\right)} \frac{\Gamma\left(2k + \mu + n + 1\right)}{\Gamma\left(2k + \mu + 1\right)} 2^{2k+\mu+n-1} \sqrt{\pi}.$$

Hence, equation (18) becomes
Equation (19) can be simplified using relation (3) as

\[
\frac{d^n Q_{\mu}}{dz^n} = \frac{2^n (-1)^n}{2^n} \sum_{k=0}^{\infty} \frac{\Gamma\left(\frac{\mu+n+1}{2}\right) \Gamma\left(\frac{\mu+n+2}{2}+k\right)}{\Gamma\left(\mu+\frac{3}{2}+k\right) k!} z^{-2^{k}+\mu+n+1}.
\]  

(19)

Again using equation (3) by taking \(z = \frac{\mu+n+1}{2}\) leads to

\[
\frac{d^n Q_{\mu}}{dz^n} = \frac{2^n (-1)^n}{2^n} \sum_{k=0}^{\infty} \frac{\Gamma\left(\frac{\mu+n+1}{2}\right) \Gamma\left(\frac{\mu+n+2}{2}+k\right)}{\Gamma\left(\mu+\frac{3}{2}+k\right) k!} z^{2k}.
\]

Hence, one has the hyper geometric representation of \(Q_{\mu}(z)\) as

\[
Q_{\mu}(z) = e^{-\sqrt{\pi} \Gamma\left(\frac{\mu+n+1}{2}\right)} z^{\frac{\mu+n+1}{2}} F\left(\frac{\mu+\frac{3}{2}+1}{2},\frac{\mu+\frac{3}{2}}{2};\frac{1}{2},\frac{\mu+n+2}{\mu+n+1}z^{-1}\right),
\]

\(\Gamma > 1, n = 0,1,2,...,\mu+\frac{3}{2}+1,0,-1,-2,...\)

(20)

Further hyper geometric representations of the second kind associated Legendre functions can be derived in a similar manner and can be found in the references (Abramowitz and Stegun 1968; Rainville 1960; Lebedev 1965; Rainville 1964; Erdelyi et al. 1953-55; Gradshteyn and Ryzhik 2007).

**Discussion**

We know that the associated Legendre functions of both kind sure defined for values of the complex variable \(z\) lie in the complement of the segment (-\(\infty\), 1]. Section 4 presents different forms of the hyper geometric representation of the associated Legendre functions of the first kind \(P_{\mu}^n(z)\) which were obtained by means of linear and quadratic transformation of the hyper geometric function and being away from any integral representation. For example, the hyper geometric representation provided by equation (12) gives the analytic continuation \(P_{\mu}^n(z)\) of \(P_{\mu}^n(z)\) into the complex region \(|1-z| < 2\), with a cut is made along the real axis from the point \(z=-\infty\) to the point \(z=+1\), whereas the hyper geometric representation given by equation (14) analytically continued into the region with the same cut mentioned above. Furthermore, the hyper geometric representation given by equation (16) carries out the analytic continuation of \(P_{\mu}^n(z)\) into the complex region \(|1-z^2| < 1\). Thus we obtain different representations of \(P_{\mu}^n(z)\) which are defined in different parts of the complex z-plane. Similarly, section 5 presents the hyper geometric representation of the associated Legendre functions of the second kind \(Q_{\mu}^n(z)\) which analytically continued \(Q_{\mu}^n(z)\) into the domain \(|z| > 1\) as shown by equation (20). By means of the hyper geometric series, such functions can be continued analytically to other regions in the complex z-plane. The function \(Q_{\mu}^n(z)\) for \(|z| > 1\) is uniquely determined by equation (20) everywhere in the z-plane in which a cut is made along the real axis from the point \(z = -\infty\) to the point \(z = +1\).
Conclusion

It was observed that the associated Legendre functions can be expressed by the hyper geometric series in suitably restricted regions of the complex z-plane cut along the real segment \((-\infty, +1]\). By rewriting the associated Legendre functions in terms of the hyper geometric function, more regions in the complex z-plane were obtained for the analytic continuation. Therefore, to conclude it is very informative to express the associated Legendre functions in terms of the hyper geometric function as shown in the discussion. It is worth to emphasize that the hyper geometric representation enables us to develop the theory of spherical functions by implementing the general theory of the hyper geometric function. Specifically, this approach is very helpful to gain the generalization of the spherical functions for arbitrary values of the degree n. It is remarkable to note that the regions of validity so often pass through a singular point of the differential equation where the regular singular points of the hyper geometric differential equation are \(z = 0, 1, \infty\). To sum up, one could claim that the linear and quadratic transformations approach of obtaining the hyper geometric representation is more convenient and less complicated than the integral representation approach.

References


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